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A "HIDDEN" SYMMETRY AND SOME OF ITS IMPLICATIONS

ΒY

VLAD GHIZDOVĂŢ¹, MIHAI MARIUS GUŢU^{1,*}, IULIANA OPREA² and CIPRIANA ŞTEFĂNESCU¹

¹"Grigore T. Popa" University of Medicine and Pharmacy, Faculty of Medicine, Biophysics and Medical Physics Department, Iaşi, Romania ²Colorado State University, Department of Mathematics, Weber Bldg. 123, Fort Collins, CO 80523, USA

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Abstract. In Physics, hidden symmetries can either be employed to describe special systems and configurations, or even be used as a basic block of a theory itself. In this paper, a special hidden symmetry of a physical field propagation phenomenon, having strong implications in the interactions theories (a classical Fresnel-type image of light propagation) is analyzed.

Keywords: hidden symmetry; propagation phenomenon; interactions; Fresnel-type light propagation.

1. Introduction

The role that symmetries play in Physics can be quite different, depending on the situation. For example, they can either describe special systems and/or configurations, or they can be a cornerstone of a theory itself (Cariglia, 2014). For this last case, we can give some examples: the Standard Model of particle physics and the General Relativity Theory, with their supersymmetric extensions. Another example is string theory, which displays a high number of symmetries and dualities. The concept of symmetry can be

^{*}Corresponding author; e-mail: mihai.gutu@umfiasi.ro

applied to various domains, such as: relativistic and non-relativistic theories, classical and quantum physics etc. Symmetries have been successfully employed in Physics to such a degree that, nowadays, a more refined strategy is needed. This is the case of General Relativity. In General Relativity the mostly used meaning of the word symmetry is associated to that of isometry, *i.e.*, a spacetime diffeomorphism that leaves the metric invariant. A one-parameter continuous isometry can be put into relation with the existence of Killing vectors. Hence, the activity in the area related to using symmetries to solve Einstein's equations or the equations of motion of other systems has been focused on finding metrics admitting Killing vectors. This activity has probably already reached its maturity. However, other types of symmetries could be used. Instead of looking at the symmetries of a spacetime, we can take into consideration a physical system evolving in a given spacetime, and thus the symmetries of the dynamics of this system can be analyzed. In this context, symmetries of the dynamics, for a classical system, involve transformations in the whole phase space of the system such that the dynamics is left invariant. Instead, for a quantum system, symmetries mean a set of phase space operators that commute with the Hamiltonian or with the relevant evolution operator, and transform solutions into solutions (Cariglia, 2014). In literature, such symmetries are often referred to as hidden symmetries (Krtouš et al., 2008; Frolov, 2008).

In the present paper, we highlight a special hidden symmetry of a physical field propagation phenomenon, having strong implications in electromagnetic and gravito-electromagnetic-type interactions. In a totally particular case, a classical Fresnel-type image of light propagation is obtained.

2. Mathematical Model

Let us consider the propagation equations for a physical field, in the compact form (Jackson, 1992; Lechner, 2018):

$$\partial_{j}\partial^{l}Q^{i} - \alpha^{2}\partial_{t}\partial^{t}Q^{i} = 0 \tag{1}$$

where

$$\partial_I \partial^I = \frac{\partial}{\partial x^I} \left(\frac{\partial}{\partial x^I} \right), \ \partial_t \partial^t = \frac{\partial^2}{\partial t^2}, \ I = 1, 2, 3$$
 (2)

and *l* is the index of summation. In the above relations, x^l is the spatial "*l*" coordinate, *t* is the time coordinate, Q^i is the "*i*" component of the physical field, and α^2 is the inverse square of the physical field propagation velocity in a material medium.

From (1), by employing the variables separation method in the form (Nagle *et al.*, 2018):

$$Q^{i}\left(x^{l},t\right) = X\left(x^{l}\right)T\left(t\right)$$
(3)

the differential equations systems is obtained:

$$\frac{\Delta X}{X} = \frac{\alpha^2}{T} \frac{d^2 T}{dt^2} = -\lambda^2 \tag{4}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$
(5)

and λ is the variables separation constant.

In this context, we will consider only the differential equation from system (4)

$$\frac{d^2T}{dt^2} + \Omega^2 T = 0 \tag{6}$$

where

$$\Omega^2 = \left(\frac{\lambda}{\alpha}\right)^2 \tag{7}$$

An operational procedure, similar with the one that follows, is also functional in the case of the differential equation:

$$\Delta X + \lambda^2 X = 0$$

The solution of (6) can be put in the form (Mercheş and Agop, 2016; Agop and Păun, 2017; Mazilu *et al.*, 2021; Mazilu and Agop, 2012):

$$T(t) = he^{i(\Omega t + \theta)} + \overline{h}e^{-i(\Omega t + \theta)}$$
(8)

where *h* is the complex amplitude, \overline{h} is the complex conjugate of *h* and θ is a phase. Thus, h, \overline{h} and θ label each entity of a material medium that has, as a "fundamental property", the same Ω .

A "hidden" symmetry, induced by a homographic group, can be seen in Eq. (6). In such a context, the ratio ε of the two independent linear solution of (6) satisfies the standard Schwartz differential equation (Mercheş and Agop, 2016; Agop and Păun, 2017; Mazilu *et al.*, 2021; Mazilu and Agop, 2012):

$$\left\{\varepsilon,t\right\} = \frac{d}{dt} \left(\frac{\ddot{\varepsilon}}{\dot{\varepsilon}}\right)^2 - \frac{1}{2} \left(\frac{\ddot{\varepsilon}}{\dot{\varepsilon}}\right)^2 = 2\Omega^2 \tag{9}$$

$$\dot{\varepsilon} = \frac{d\varepsilon}{dt}, \quad \ddot{\varepsilon} = \frac{d^2\varepsilon}{dt^2}$$
 (10)

The left part of (9) is invariant with respect to the homographic transformations

$$\varepsilon \leftrightarrow \varepsilon' = \frac{a\varepsilon + b}{c\varepsilon + d} \tag{11}$$

with a, b, c and d and real parameters. Relation (11), corresponding to all possible values of these parameters, defines the group SL(2R) (Lang, 2011).

Thus, a "personal" parameter ε for each material medium entity can be constructed. Indeed, the solution given by (9) becomes a "guide" that can be written as

$$\varepsilon = l + m \tan\left(\Omega t + \theta\right) \tag{12}$$

It can now be seen that, through l, m and θ , it is possible to characterize any material medium entity. Furthermore, by identifying the phase from (12) with the one from (8), the "personal" parameter becomes:

$$\varepsilon' = \frac{h + \overline{h}\varepsilon}{1 + h}, \quad h = l + im, \quad \overline{h} = l - im, \quad \varepsilon \equiv e^{2i(\Omega t + \theta)}$$
 (13)

The fact that (12) is also a solution of (9) implies, by explicating (11), the SL(2R) group (Mercheş and Agop, 2016; Agop and Păun, 2017; Mazilu *et al.*, 2021; Mazilu and Agop, 2012):

$$h = \frac{ah+b}{ch+d}, \quad \overline{h} = \frac{a\overline{h}+b}{c\overline{h}+d}, \quad \varepsilon = \frac{c\overline{h}+d}{ch+d}\varepsilon$$
 (14)

Therefore, group (14) works as "synchronization modes" among the entities of any material medium.

The structure of group (14) is a typical SL(2R) one:

$$[A_1, A_2] = A_1, \ [A_2, A_3] = A_3, \ [A_3, A_1] = -2A_2$$
(15)

In the previous relations, A_k , k = 1, 2, 3 are the infinitesimal generators of the group. Since the group is simple transitive, A_k are the components of the Cartan coframe, in the form (Crampin, 2016)

$$d(f) = \sum \frac{\partial f}{\partial x^{k}} dx^{k} = \begin{cases} \omega^{1} \left[h^{2} \frac{\partial}{\partial h} + \overline{h}^{2} \frac{\partial}{\partial \overline{h}} + (h - \overline{h}) \varepsilon \frac{\partial}{\partial \varepsilon} \right] + \\ + 2\omega^{2} \left(h \frac{\partial}{\partial h} + \overline{h} \frac{\partial}{\partial \overline{h}} \right) + \omega^{3} \left(\frac{\partial}{\partial h} + h \frac{\partial}{\partial \overline{h}} \right) \end{cases} (f) \quad (16)$$

with ω^k the components of the Cartan coframe:

$$dh = \omega^{1}h^{2} + 2\omega^{2}h + \omega^{3}, \ d\overline{h} = \omega^{1}\overline{h}^{2} + 2\omega^{2}\overline{h} + \omega^{3}, \ d\varepsilon = \omega^{1}\varepsilon\left(h - \overline{h}\right)$$
(17)

Thus, we can immediately obtain the infinitesimal generators and the coframe by identifying the right-hand side of (16) with the standard dot product of the SL(2R) algebra

$$\omega^1 A_3 + \omega^3 A_1 - 2\omega^2 A_2 \tag{18}$$

so that

$$A_{1} = \frac{\partial}{\partial h} + \frac{\partial}{\partial \overline{h}}, A_{2} = h \frac{\partial}{\partial h} + \overline{h} \frac{\partial}{\partial \overline{h}}, A_{3} = h^{2} \frac{\partial}{\partial h} + \overline{h}^{2} \frac{\partial}{\partial \overline{h}} + \left(h - \overline{h}\right) \varepsilon \frac{\partial}{\partial \varepsilon}$$
(19)

and

$$\omega^{1} = \frac{dk}{\left(h - \overline{h}\right)k}, \ 2\omega^{2} = \frac{dh - d\overline{h}}{h - \overline{h}} - \frac{h + \overline{h}}{h - \overline{h}}\frac{d\varepsilon}{\varepsilon}, \ \omega^{3} = \frac{hdh - \overline{h}dh}{h - \overline{h}} + \frac{h\overline{h}d\varepsilon}{\left(h - \overline{h}\right)\varepsilon}$$
(20)

In real terms from (13), these last equations can be written as

$$A_{1} = \frac{\partial}{\partial l}, A_{2} = l\frac{\partial}{\partial l} + m\frac{\partial}{\partial m}, A_{3} = \left(l^{2} - m^{2}\right)\frac{\partial}{\partial l} + 2lm\frac{\partial}{\partial m} + 2m\frac{\partial}{\partial \theta}$$
(21)

$$\omega^{1} = \frac{d\theta}{2m}, \ \omega^{2} = \frac{dm}{m} - \frac{l}{m}d\theta, \ \omega^{3} = \frac{l^{2} + m^{2}}{2m}d\theta + \frac{mdl - ldm}{m}$$
(22)

It should be mentioned that, in (Barbilian, 1935), the absolute invariant differentials are used:

$$\omega^{1} = \frac{dh}{\left(h - \overline{h}\right)}, \ \omega^{2} = -i\left(\frac{d\varepsilon}{\varepsilon} - \frac{dh + d\overline{h}}{h - \overline{h}}\right), \ \omega^{3} = \frac{\varepsilon d\overline{h}}{h - \overline{h}}$$
(23)

or, in real terms, exhibiting a three - dimensional Lorentz structure of this space

$$\Omega^{1} = \omega^{1} = d\theta + \frac{dl}{m}, \ \Omega^{2} = \cos\theta \frac{dl}{m} + \sin\theta \frac{dm}{m}, \ \Omega^{3} = c - \sin\theta \frac{dl}{m} + \cos\theta \frac{dm}{m}$$
(24)

In this context, a connection with the Poincaré representation of the Lobachevsky plane can be found. Indeed, the metric here is:

$$\frac{ds^2}{g} = \left(\omega^2\right)^2 - 4\omega^1 \omega^2 = \left(\frac{d\varepsilon}{\varepsilon} - \frac{dh + d\overline{h}}{h - \overline{h}}\right)^2 + 4\frac{dhd\overline{h}}{\left(h - \overline{h}\right)^2}$$
(25)

or in real terms

$$-\frac{ds^2}{g} = -\left(\Omega^1\right)^2 + \left(\Omega^2\right)^2 + \left(\Omega^3\right)^2 = -\left(d\theta + \frac{dl}{m}\right) + \frac{dl^2 + dm^2}{m^2}$$
(26)

where *g* is a constant.

These metrics reduce to that of Poincaré in cases when $\omega^2 = 0$ or $\Omega^1 = 0$ which defines the variable θ as the "angle of parallelism" (in Levi-Civita sense) of the hyperbolic plane (the connection). In fact, if we recall that in modern terms dl/dm represents the connection form of the hyperbolic plane, relations (24) then represent a general Bäcklung transformation in that plane (Darling, 1994).

The symmetry written above highlights the form:

$$y_1 = \frac{h + \overline{h}\varepsilon}{1 + \varepsilon}, \ y_2 = \frac{h + w\overline{h}\varepsilon}{1 + w\varepsilon}, \ y_3 = \frac{h + w^2\overline{h}\varepsilon}{1 + w^2\varepsilon}$$
 (27)

of the real roots of the cubic:

$$a_0 y^3 + 3a_1 y^2 + 3a_2 y + a_3 = 0$$

(28)
$$a_0, a_1, a_2, a_3 \in \mathbb{R}$$

where h and \overline{h} are the roots of Hessian

$$\left(a_{0}a_{2}-a_{1}^{2}\right)y^{2}+\left(a_{0}a_{3}-a_{1}a_{2}\right)y+\left(a_{1}a_{3}-a_{2}^{2}\right)=0$$
(29)

and $w = (-1 + i\sqrt{3})/2$ is the cubic root of the unit, different from the unit itself (Mercheş and Agop, 2016; Agop and Păun, 2017; Mazilu *et al.*, 2021; Mazilu and Agop, 2012). Taking into account that relations (27) can always be put into relation with the eigenvalues of a second order tensor, it results that this tensor could characterize the physics of the phenomenon described by the differential (1). To this end, let us admit that this tensor w_{ij} has the form (Mazilu and Agop, 2012):

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$$W_{ij} = \lambda u_{ij} + \mu V_{ij} \tag{30}$$

where λ and μ are real parameters describing the extent to which the analyzed physical phenomenon is "spatial" and, respectively, "material". The matrices **u** and **v** can be defined through

$$u_{ij} = u_i u_j - \frac{1}{2} u^2 \delta_{ij}; v_{ij} = v_i v_j - \frac{1}{2} v^2 \delta_{ij}$$

$$u^2 = u_1^2 + u_2^2 + u_3^2; v^2 = v_1^2 + v_2^2 + v_3^2$$
(31)

where δ_{ii} is Kronecker's pseudo-tensor. When detailed, matrix (30) becomes

$$w_{ij} = \lambda u_i u_j + \mu v_i v_j - \frac{1}{2} (\lambda u^2 + \mu v^2) \delta_{ij}$$
(32)

We want to highlight now that this tensor has three main real and distinct eigenvalues. Indeed, its orthogonal invariants are

$$I_1 = -e; I_2 = -e^2 + g^2; I_3 = -e(e^2 - g^2)$$
 (33)

where we have used

$$e = \frac{1}{2} (\lambda u^2 + \mu v^2); \ \vec{g} = \sqrt{\lambda \mu} (\vec{u} \times \vec{v})$$
(34)

Using the standard approach, the main values of the tensor w_{ij} can be obtained as roots of the matrix's characteristic equation:

$$W_1 = e, \ W_{2,3} = \pm \sqrt{e^2 - g^2}$$
 (35)

The pair from equation (34) is one of the own vectors of **w**, together with its own value. The other two own vectors of **w** are perpendicular, and they are located in the planes of vectors \vec{u} and \vec{v} .

The magnitudes

$$w_n = \frac{w_1 + w_2 + w_3}{3}, \ w_t^2 = \frac{1}{15} [(w_2 - w_3)^2 + (w_3 - w_1)^2 + (w_1 - w_2)^2]$$
 (36)

are the Novojilov averages for the normal and shearing components of the **w** tensor in any given point (Novojilov, 1952).

In the following, let us define the vector formed by the \mathbf{w} matrix eigenvalues:

$$|w\rangle \equiv \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$
(37)

Now, by choosing the octahedral plane with a normal given by the unitary vector

$$|n\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$
(38)

the normal component w_n on this plane is

$$\left\langle n \middle| w \right\rangle = \frac{w_1 + w_2 + w_3}{3} \tag{39}$$

Another quantity in (36) is the vector's norm:

$$|w_{t}\rangle \equiv |w\rangle - |n\rangle\langle n|w\rangle = \frac{1}{3} \begin{pmatrix} 2w_{1} - w_{2} - w_{3} \\ -w_{1} + 2w_{2} - w_{3} \\ -w_{1} - w_{2} + 2w_{3} \end{pmatrix}$$
(40)

After a simple calculation, we obtain:

$$\left\langle W_t \left| W_t \right\rangle \equiv \frac{1}{5} W_t^2 \tag{41}$$

For a particular case of eigenvalues (35), the two magnitudes can be written:

$$w_{n} = \langle w | n \rangle = -\frac{2}{\sqrt{3}}e, |w_{t}\rangle = \frac{2}{3} \begin{pmatrix} -2e \\ 3\sqrt{e^{2} - \vec{g}^{2}} + e \\ -3\sqrt{e^{2} - \vec{g}^{2}} + e \end{pmatrix}$$
(42)

In this context, taking into account the previous relations, the following calculations referring to the normal and shearing component in a particular point in space, are obtained:

$$\langle \xi | n \rangle = -\frac{1}{\sqrt{3}} \xi^2, \quad |\xi_t\rangle = \frac{2}{3} \xi^2 \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$
 (43)

When vector $\vec{\xi}$ is perpendicular both on \vec{u} and on \vec{v} , the tensors **w** and ξ commute. In this case, the direction of the vector in (43) is fixed and can be taken as reference in the octahedral plane. The angle ψ of the vector in equation (42) is given by:

$$\cos\psi = -\frac{e}{\sqrt{4e^2 - 3\vec{g}^2}} \tag{44}$$

It results that, in these conditions, ψ is independent of the reference vector, and depends only on the description of the physical propagation phenomenon. By appropriately choosing a sign of the square root in the denominator of this formula, the origin $\psi = 0$ of that angle appears at e = g. In its turn, this condition means that the angle θ between vectors \vec{u} and \vec{v} is given by the equation

$$\left|\sin\theta\right| = \frac{1}{2} \frac{\left|\lambda u^2 + \mu v^2\right|}{\sqrt{\lambda \mu u v}}$$
(45)

As the quantity on the right-hand side of this equation is always greater or equal to one, the angle between vectors \vec{u} and \vec{v} can only be 90°. Therefore, the initial condition of the w tensor in the octahedral plane translates into the fact that vectors \vec{u} and \vec{v} are perpendicular on each other, and their plane is perpendicular on vector ξ . In a particular case, if this last vector is given by the direction of a light beam, we can obtain the classic image of light propagation according to Fresnel (Fresnel, 1827).

3. Conclusions

In this paper, we developed original methodologies, based on operational procedures (group invariances, variational principles etc.), for describing the dynamics of physical systems.

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O SIMETRIE "ASCUNSĂ" ȘI UNELE IMPLICAȚII

(Rezumat)

Simetriile "ascunse" pot fi folosite în fizică pentru a descrie sisteme și configurații speciale, sau chiar pentru a construi "temelia" unei noi teorii. În prezenta lucrare se analizează o simetrie ascunsă a unui fenomen de propagare a unui câmp fizic, ce are implicații puternice în teoriile de interacții (o imagine clasică de tip Fresnel a propagării luminii).